MS221 Exploring Mathematics

Block A Tutorial Resource Material

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MS221 Chapter A1

Activity 1.1

Using the properties of the roots of quadratic equations, show that the roots ϕ and ψ of the equation $x^2 - x - 1 = 0$ satisfy the following identities:

(a) $\phi + \psi = 1$, (b) $\phi\psi = -1$, (c) $\phi^2 = \phi + 1$, (d) $\phi = 1 + 1/\phi$,

Activity 1.2

(a) Using $\phi^2 = \phi + 1$ twice, show that $\phi^3 = 2\phi + 1$.

(b) Hence find expressions for ϕ^4, ϕ^5 and ϕ^6.

(c) Use your results to conjecture a general formula for ϕ^n in terms of ϕ and Fibonacci numbers.

(d) Use your calculator to confirm that your conjecture holds for $n = 10$.

Activity 1.3

(a) Given that the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β,
find in terms of a and b the quadratic equations, the quadratic equations whose roots are:
(i) $1/\alpha$ and $1/\beta$, (ii) α^2 and β^2, (iii) α^3 and β^3.

(b) Show that $\alpha = \frac{1}{3}$ and $\beta = -2$ satisfy the equation $3x^2 + 5x - 2 = 0$. Use these facts to confirm your answers to part (a) for these values of α and β.

Activity 1.4

(a) (i) By rationalising the denominator, show that $1/\phi$ and $1/\psi$ are $-\psi$ and $-\phi$ respectively.
(ii) Hence or otherwise form the equation with roots $1/\phi$ and $1/\psi$.

(b) (i) Using only variations of the identities in question 1 above and their equivalents in terms of ψ, form the quadratic equation whose roots are ϕ^2 and ψ^2.
(ii) Solve the equation and verify that its roots are ϕ^2 and ψ^2.

Activity 1.5

(a) (i) Write down the first 9 Fibonacci numbers. (See table on page 17 of the chapter.)
(ii) Calculate $F_2^2 - F_1^2$, $F_3^2 - F_2^2$, $F_4^2 - F_3^2$, $F_5^2 - F_4^2$, $F_6^2 - F_5^2$ and $F_7^2 - F_6^2$.

(b) How are your answers related to the first eight Fibonacci numbers?

(c) Conjecture a relationship between any four consecutive Fibonacci numbers and prove your conjecture.

Activity 1.6

(a) Use the Fibonacci sum identity to calculate the sum of the Fibonacci numbers
$F_{n+1} + F_{n+2} + \ldots F_{2n}$ in terms of Fibonacci numbers.

(b) Confirm your answer by calculating the sum when $n = 3$.

(c) Hence find the mean of the Fibonacci numbers $F_{n+1}, F_{n+2}, \ldots F_{2n}$

Activity 1.7

Write down the next 2 terms and find the closed form for each of the following second order linear recurrence sequences. Use the closed form to confirm your answer for u_3.

(a) $u_0 = 5,\ u_1 = 10,\quad u_{n+2} = u_{n+1} + 6u_n\ (n = 0,1,2,\ldots)$

(b) $u_0 = 4,\ u_1 = 7,\quad u_{n+2} = 5u_{n+1} - 6u_n\ (n = 0,1,2,\ldots)$

(c) $u_0 = 3,\ u_1 = -14,\quad u_{n+2} = -4u_{n+1} - 4u_n\ (n = 0,1,2,\ldots)$

Activity 1.8

Show by substitution that in each of the following examples, the sequence satisfies the given recurrence system.

(a) Sequence: $u_n = 4^{n-1} + 3^{n-1}$ $(n = 1, 2, 3 \ldots)$;
recurrence system: $u_1 = 2$, $u_2 = 7$, $u_{n+2} = 7u_{n+1} - 12u_n$.

(b) Sequence: $u_n = 4^n + (-3)^n$ $(n = 0, 1, 2\ldots)$;
recurrence system: $u_0 = 2$, $u_1 = 1$, $u_{n+2} = u_{n+1} + 12u_n$.

MS221 Chapter A2

Activity 2.1

For each of the following equations, identify the conic it represents and find, where appropriate, the centre, eccentricity, focus (or foci), directrix (or directrices) and asymptotes of the curve. In each case, sketch the curve and mark on the points or lines you have identified.

(a) $\frac{x^2}{25}+\frac{y^2}{4}=1$ (b) $x^2+y^2=4$ (c) $y^2=6x$ (d) $\frac{x^2}{2}-2y^2=1$

Activity 2.2

Explain why the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$ does not represent a conic in standard position.

Activity 2.3

(a) An ellipse in standard position has a major axis of length 6 and a minor axis of length 4. Find its equation, eccentricity, the coordinates of its foci and the equations of its directrices. Sketch the curve and mark on the points or lines you have identified.

(b) An hyperbola in standard position has vertices at $(2,0)$ and $(-2,0)$ and asymptotes $3x-4y=0$ and $3x+4y=0$. Find its equation, eccentricity, the coordinates of its foci and the equations of its directrices.

Activity 2.4

For each of the following, determine what type of conic it is and sketch it.

(a) The conic with equation $9x^2-4y^2=1$.

(b) The conic with equation $x^2+12y^2=3$.

(c) The conic in standard position with focus at $(4\sqrt{2},\,0)$ and directrix $x=2\sqrt{2}$.

(d) The conic in standard position with eccentricity $e=\sqrt{3}/2$ and directrix $x=2\sqrt{3}$.

Activity 2.5

By completing the square, identify each of the following curves and sketch it.

(a) $9x^2+54x+4y^2-16y+61=0$

(b) $9x^2-36x-25y^2-150y-414=0$

Activity 2.6

This question concerns the parabola K with equation $y^2=8x$.

(a) Sketch the curve, and mark on its focus and directrix.

(b) Insert the line through the focus and at right angles to the axis. Where does this line meet the curve? What is the distance between the two points of intersection?

(c) Confirm that the point A with coordinates $(\frac{1}{2},2)$ lies on the curve and show that it satisfies the focus–directrix property for the parabola.

(d) Express the coordinates of any point on the curve in parametric form in terms of the parameter t. What value of t corresponds to the point A?

MS221 Chapter A3

Activity 3.1

The curve K with equation $y^2 = 8x$ translated to become L ,with its vertex at $(3, -1)$.

(a) Write down the translation function which maps K onto L.

(b) What is the equation of L? What is the equation of its axis?

(c) Sketch the curve, and mark on its focus and directrix.

(d) The point A with coordinates $(\frac{1}{2},2)$ lies on the curve $y^2 = 8x$. Find the coordinates of A', the image of A under the translation and confirm that it satisfies the focus–directrix property for the translated parabola.

(e) Express the coordinates of any point on the translated curve in terms of the parameter t. What value of t corresponds to the point A'?

(f) By eliminating the parameter t, find the equation of the transformed graph, and confirm it agrees with your answer to (a) above.

Activity 3.2

Write down a parametrisation function for each of the following curves. In each case, check your answer by finding the coordinates of two points on the original curve, and finding the appropriate value for t that results in those coordinates.

(a) $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ (b) $\dfrac{(x-3)^2}{8} + \dfrac{(y+4)^2}{6} = 1$ (c) $x^2 + y^2 + 4x - 6y - 3 = 0$

Activity 3.3

(a) Find the image of the general point (x, y) under the following isometries:

(i) $t_{3,-4}$ (ii) $r_{\pi/6}$ (iii) $q_{\pi/3}$ (iv) $r_{\pi/6} \circ q_{\pi/3}$

(b) Hence find the images of the following points under the transformations above:

$(2, 1)$, $(2,-1)$, $(-2, 1)$ and $(-2, -1)$. Give decimals correct to 2 d.p.

(c) Verify that your answers conform to an isometric transformation by sketching the original rectangle and its image under each of the transformations above.

Activity 3.4

Use the formulas in the Handbook:

(a) to prove the following identities

(i) $(\cot\theta - \operatorname{cosec}\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$ (ii) $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = 2\operatorname{cosec}2\theta$

(b) and without using a calculator, to show that:

(i) $2\cos^2(\pi/8) = 1 + 1/\sqrt{2}$ (ii) $\tan 15^\circ = 2 - \sqrt{3}$ (iii) $\sin 75^\circ = \dfrac{\sqrt{2}}{4}(1+\sqrt{3})$

Activity 3.5

Use the strategy in the Handbook to sketch the curve L with equation:

$35x^2 - 8\sqrt{5}xy + 37y^2 - 135 = 0$